Ronald H. Dieck. "Measurement Accuracy."

Measurement Accuracy

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All test measurements are taken so that data may be acquired that are useful in decision making. No tests are run and no measurements made when the "answer" is already known. For data to be useful, it is necessary that their measurement errors be small in comparison to the changes or effect under evaluation. Measurement error is unknown and unknowable. This chapter addresses the techniques used to estimate, with some confidence, the expected limits of the measurement errors.

4.1 Error: The Normal Distribution and the Uniform Distribution

Error is defined as the difference between the measured value and the true value of the meaurand [1]. That is,

$$E = \left(measured\right) - \left(true\right) \tag{4.1}$$

where E = the measurement error
(measured) = the value obtained by a measurement
(true) = the true value of the measurand

It is only possible to estimate, with some confidence, the expected limits of error. The most common method for estimating those limits is to use the *normal distribution* [2]. It is

$$Y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(X-\mu\right)^2/\sigma^2} \tag{4.2}$$

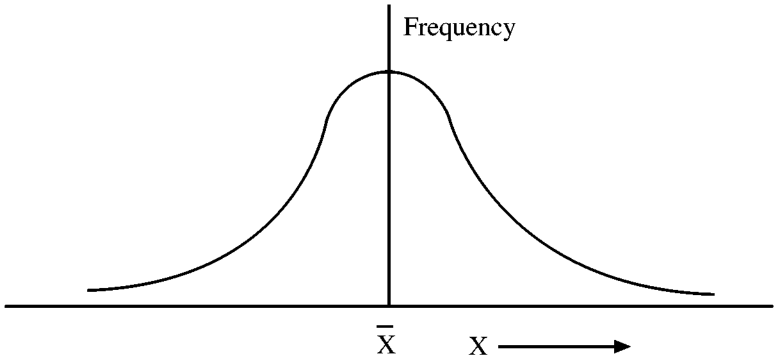

FIGURE 4.1

where X = the input variable, here the value obtained by a measurement
μ = the average of the population of the X variable
σ = the standard deviation of the population, expressed as:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \mu\right)^2}{n}} \tag{4.3}$$

where X_i = the i^{th} X measurement
n = the number of data points measured from the population

Typically, neither n, μ, nor σ are known.

Figure 4.1 illustrates this distribution. Here, for an infinite population ($N = \infty$), the standard deviation, σ, would be used to estimate the expected limits of a particular error with some confidence. That is, the average, plus or minus 2σ divided by the square root of the number of data points, would contain the true average, μ, 95% of the time.

However, in test measurements, one typically cannot sample the entire population and must make do with a sample of N data points. The sample standard deviation, S_X, is then used to estimate σ_X, the expected limits of a particular error. (That sample standard deviation divided by the square root of the number of data points is the starting point for the confidence interval estimate on μ.) For a large dataset (defined as having 30 or more degrees of freedom), plus or minus $2S_X$ divided by the square root of the number of data points in the reported average, M, would contain the true average, μ, 95% of the time. That S_X divided by the square root of the number of data points in the reported average is called the *standard deviation of the average* and is written as:

$$S_{\bar{X}} = \sqrt{\frac{\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2}{N-1}} \Bigg/ \sqrt{M} = S_X \quad \sqrt{M} \tag{4.4}$$

where $S_{\bar{X}}$ = the standard deviation of the average; the sample standard deviation of the data divided by the square root of M
S_X = the sample standard deviation
$\bar{X}$ = the sample average, that is,

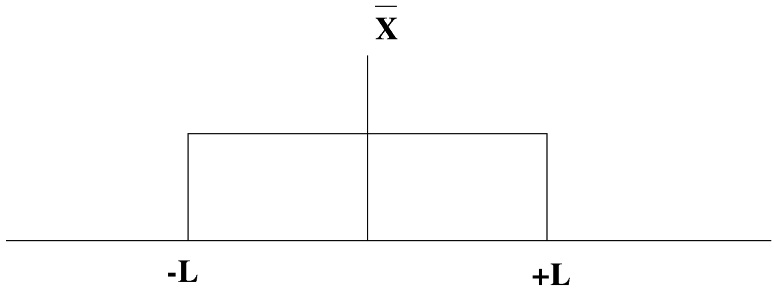

FIGURE 4.2

$\bar{X}$ $= \sum_{i=1}^{M} \left(X_i / N \right)$ (4.5)

X_i = the i^{th} data point used to calculate the sample standard deviation and the average, $\bar{X}$, from the data

N = the number of data points used to calculate the standard deviation

$(N-1)$ = the degrees of freedom of S_X and $S_{\bar{X}}$

M = the number of data points in the reported average test result

Note in Equation 4.4 that N does not necessarily equal M. It is possible to obtain S_X from historical data with many degrees of freedom ([$N-1$] greater than 30) and to run the test only M times. The test result, or average, would therefore be based on M measurements, and the standard deviation of the average would still be calculated with Equation 4.4. In that case, there would be two averages, $\bar{X}$. One $\bar{X}$ would be from the historical data used to calculate the sample standard deviation, and the other $\bar{X}$, the average test result for M measurements.

Note that the sample standard deviation, S_X, is simply:

$$S_X = \sqrt{\frac{\sum_{i=1}^{N} \left(X_i - \bar{X} \right)^2}{N-1}} \quad (4.6)$$

In some cases, a particular error distribution may be assumed or known to be *a uniform or rectangular distribution*, Figure 4.2, instead of a normal distribution. For those cases, the sample standard deviation of the data is calculated as:

$$S_X = L / \sqrt{3} \quad (4.7)$$

where L = the plus/minus limits of the uniform distribution for a particular error [3].

For those cases, the standard deviation of the average is written as:

$$S_{\bar{X}} = \frac{L/\sqrt{3}}{\sqrt{M}} \quad (4.8)$$

Although the calculation of the sample standard deviation (or its estimation by some other process) is required for measurement uncertainty analysis, all the analytical work computing the measurement uncertainty uses only the standard deviation of the average for each error source.

Uncertainty (Accuracy)

Since the error for any particular error source is unknown and unknowable, its limits, at a given confidence, must be estimated. This estimate is called the *uncertainty.* Sometimes, the term *accuracy* is used to describe the quality of test data. This is the positive statement of the expected limits of the data's errors. Uncertainty is the negative statement. Uncertainty is, however, unambiguous. Accuracy is sometimes ambiguous. (For example,: what is twice the accuracy of ±2%? ±1% or ±4%?) For this reason, this chapter will use the term *uncertainty* throughout to describe the quality of test data.

4.2 Measurement Uncertainty Model

Purpose

One needs an estimate of the uncertainty of test results to make informed decisions. Ideally, the uncertainty of a well-run experiment will be much less than the change or test result expected. In this way, it will be known, with high confidence, that the change or result observed is real or acceptable and not a result of the errors of the test or measurement process. The limits of those errors are estimated with uncertainty, and those error sources and their limit estimators, the uncertainties, may be grouped into classifications to ease their understanding.

Classifying Error and Uncertainty Sources

There are two classification systems in use. The final total uncertainty calculated at a confidence is identical no matter what classification system is used. The two classifications utilized are the *ISO classifications* and the *engineering classifications.* The former groups errors and their uncertainties by type, depending on whether or not there is data available to calculate the sample standard deviation for a particular error and its uncertainty. The latter classification groups errors and their uncertainties by their effect on the experiment or test. That is, the engineering classification groups errors and uncertainties by *random* and *systematic* types, with subscripts used to denote whether there are data to calculate a standard deviation or not for a particular error or uncertainty source. For this reason, engineering classification groups usually are more useful and recommended.

ISO Classifications

This error and uncertainty classification system is not recommended in this chapter, but will yield a total uncertainty in complete agreement with the recommended classification system — the engineering classification system. In this ISO system, errors and uncertainties are classified as Type A if there are data to calculate a sample standard deviation and Type B if there is not [4]. In the latter case, the sample standard deviation might be obtained from experience or manufacturer's specifications, to name two examples.

The impact of multiple sources of error is estimated by root-sum-squaring their corresponding multiple uncertainties. The operating equations are

Type A, data for the calculation of the standard deviation:

$$U_A = \left[\sum_{i=1}^{N_A}\left(\theta_i U_{A_i}\right)^2\right]^{1/2} \tag{4.9}$$

where U_{A_i} = the standard deviation (based on data) of the average for uncertainty source i of Type A each with its own degrees of freedom. U_A is in units of the test or measurement result. It is an $S_{\bar{X}}$.

N_A = the number of parameters with a Type A uncertainty

θ_i = the sensitivity of the test or measurement result, R, to the i^{th} Type A uncertainty. θ_i is the partial derivative of the result with respect to each i^{th} independent measurement.

The uncertainty of each error source in units of that source, when multiplied by the sensitivity for that source, converts that uncertainty to result units. Then the effect of several error sources may be estimated by root-sum-squaring their uncertainties as they are now all in the same units. The sensitivities, θ_i, are obtained for a measurement result, R, which is a function of several parameters, P_i. The basic equations are

R = the measurement result

where $R = f(P_1, P_2, P_3 \ldots P_N)$

P = a measurement parameter used to calculate the result, R

$\theta_i = \partial R / \partial P_i$

Obtaining the θ_i is often called error propagation or uncertainty propagation.

Type B (no data for standard deviation) calculation

$$U_B = \left[\sum_{i=1}^{N_B} \left(\theta_i U_{B_i} \right)^2 \right]^{1/2} \tag{4.10}$$

where U_{B_i} = the standard deviation (based on an estimate, not data) of the average for uncertainty source i of Type B; U_B is in units of the test or measurement result, R. It is an $S_{\bar{X}}$

N_B = the number of parameters with a Type B uncertainty

θ_i = the sensitivity of the test or measurement result to the i^{th} Type B uncertainty R

For these uncertainties, it is assumed that the U_{B_i} represent one standard deviation of the average for one uncertainty source with an assumed normal distribution. (They also represent one standard deviation as the square root of the "M" by which they are divided is one, that is, there is only one Type B error sampled from each of these distributions.) The degrees of freedom associated with this standard deviation (also standard deviation of the average) is infinity.

Note that θ_i, the sensitivity of the test or measurement result to the i^{th} Type B uncertainty, is actually the change in the result, R, that would result from a change, of the size of the Type B uncertainty, in the i^{th} input parameter used to calculate that result.

The degrees of freedom of the U_A and the U_{B_i} are needed to compute the degrees of freedom of the combined total uncertainty. It is calculated with the Welch–Satterthwaite approximation. The general formula for degrees of freedom [5] is

$$df_R = \nu_R = \frac{\left[\sum_{i=1}^{N} \left(S_{\bar{X}_i} \right)^2 \right]^2}{\left[\sum_{i=1}^{N} \frac{\left(S_{\bar{X}_i} \right)^4}{\left(\nu_i \right)} \right]} \tag{4.11}$$

where $df_R = \nu_R$ = degrees of freedom for the result
ν_i = the degrees of freedom of the i^{th} standard deviation of the average

For the ISO model, Equation 4.11 becomes:

$$df_{R,\mathrm{ISO}} = \nu_{R,\mathrm{ISO}} = \frac{\left[\sum_{i=1}^{N_A}\left(\theta_i U_{\mathrm{A}}\right)^2 + \sum_{i=1}^{N_B}\left(\theta_i U_{\mathrm{B}_i}\right)^2\right]^2}{\left[\sum_{i=1}^{N_A}\frac{\left(\theta_i U_{\mathrm{A}_i}\right)^4}{\left(\nu_i\right)} + \sum_{i=1}^{N_B}\frac{\left(\theta_i U_{\mathrm{B}_i}\right)^4}{\left(\nu_i\right)}\right]} \tag{4.12}$$

The degrees of freedom calculated with Equation 4.12 is often a fraction. This should be truncated to the next lower whole number to be conservative.

Note that in Equations 4.9, 4.10, and 4.12, N_{A} and N_{B} need not be equal. They are only the total number of parameters with uncertainty sources of Type A and B, respectively.

In computing a total uncertainty, the uncertainties noted by Equations 4.10 and 4.11 are combined. For the ISO model [3], this is calculated as:

$$U_{R,\mathrm{ISO}} = \pm t_{95}\left[\left(U_{\mathrm{A}}\right)^2 + \left(U_{\mathrm{B}}\right)^2\right]^{1/2} \tag{4.13}$$

where t_{95} = Student's t for ν_R degrees of freedom

Student's t is obtained from Table 4.1.

Note that alternative confidences are permissible. 95% is recommended by the ASME [6], but 99% or 99.7% or any other confidence is obtained by choosing the appropriate Student's t. 95% confidence is, however, recommended for uncertainty analysis.

In all the above, the errors were assumed to be independent. Independent sources of error are those that have no relationship to each other. That is, an error in a measurement from one source cannot be used to predict the magnitude or direction of an error from the other, independent, error source. Nonindependent error sources are related. That is, if it were possible to know the error in a measurement from one source, one could calculate or predict an error magnitude and direction from the other,

TABLE 4.1 Student's t Statistic for 95% Confidence, t_{95}, Degrees of Freedom, ν. This is Frequently Written as: $t_{95,\nu}$

ν	t_{95}	ν	t_{95}	ν	t_{95}
1	12.706	11	2.201	21	2.080
2	4.303	12	2.179	22	2.074
3	3.182	13	2.160	23	2.069
4	2.776	14	2.145	24	2.064
5	2.571	15	2.131	25	2.060
6	2.447	16	2.120	26	2.056
7	2.365	17	2.110	27	2.052
8	2.306	18	2.101	28	2.048
9	2.262	19	2.093	29	2.045
10	2.228	20	2.086	≥30	2.000

nonindependent error source. These are sometimes called *dependent error sources*. Their degree of dependence may be estimated with the linear correlation coefficient. If they are nonindependent, whether Type A or Type B, Equation 4.13 becomes [7]:

$$U_{R,\mathrm{ISO}} = t_{95}\left\{\sum_{T=\mathrm{A}}^{\mathrm{B}}\sum_{i=1}^{N_{i,\mathrm{T}}}\left[\left(\theta_i U_{i,\mathrm{T}}\right)^2 + \sum_{j=1}^{N_{j,\mathrm{T}}}\theta_i\theta_j U_{(i,\mathrm{T}),(j,\mathrm{T})}\left(1-\delta_{i,j}\right)\right]\right\}^{1/2} \tag{4.14}$$

where: $U_{i,\mathrm{T}}$ = the i^{th} elemental uncertainty of Type T (can be Type A or B)

$U_{R,\mathrm{ISO}}$ = the total uncertainty of the measurement or test result

θ_i = the sensitivity of the test or measurement result to the i^{th} Type T uncertainty

θ_j = the sensitivity of the test or measurement result to the j^{th} Type T uncertainty

$U_{(i,\mathrm{T}),(j,\mathrm{T})}$ = the covariance of U_{i,T_i} on $U_{j,\mathrm{T}}$

$$= \sum_{l=1}^{K} U_{i,\mathrm{T}}(l)\,U_{j,\mathrm{T}}(l) \tag{4.15}$$

= the sum of the products of the elemental systematic uncertainties that arise from a common source (l)

l = an index or counter for common uncertainty sources

K = the number of common source pairs of uncertainties

$\delta_{i,j}$ = the Kronecker delta. $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$ if not [7]

T = an index or counter for the ISO uncertainty type, A or B

This ISO classification equation will yield the same total uncertainty as the engineering classification, but the ISO classification does not provide insight into how to improve an experiment's or test's uncertainty. That is, whether to possibly take more data because the random uncertainties are too high or calibrate better because the systematic uncertainties are too large. The engineering classification now presented is therefore the preferred approach.

Engineering Classification

The engineering classification recognizes that experiments and tests have two major types of errors whose limits are estimated with uncertainties at some chosen confidence. These error types may be grouped as *random* and *systematic*. Their corresponding limit estimators are the random uncertainty and systematic uncertainties, respectively.

Random

The general expression for random uncertainty is the ($1S_{\overline{X}}$) standard deviation of the average [6]:

$$S_{\overline{X},R} = \left[\sum_{T=\mathrm{A}}^{\mathrm{B}}\sum_{i=1}^{N_{i,\mathrm{T}}}\left(\theta_i S_{\overline{X}_{i,\mathrm{T}}}\right)^2\right]^{1/2} = \left[\sum_{T=\mathrm{A}}^{\mathrm{B}}\sum_{i=1}^{N_{i,\mathrm{T}}}\left(\theta_i S_{X_{i,\mathrm{T}}}\Big/\sqrt{M_{i,\mathrm{T}}}\right)^2\right]^{1/2} \tag{4.16}$$

where: $S_{X_{i,\mathrm{T}}}$ = the sample standard deviation of the i^{th} random error source of Type T

$S_{\overline{X}_{i,\mathrm{T}}}$ = the random uncertainty (standard deviation of the average) of the i^{th} parameter random error source of Type T

$S_{\overline{X},R}$ = the random uncertainty of the measurement or test result

$N_{i,T}$ = the total number of random uncertainties, Types A and B, combined

$M_{i,\mathrm{T}}$ = the number of data points averaged for the i^{th} error source, Type A or B

θ_i = the sensitivity of the test or measurement result to the i^{th} random uncertainty

Note that $S_{\bar{X},R}$ is in units of the test or measurement result because of the use of the sensitivities, θ_i. Here, the elemental random uncertainties have been root-sum-squared with due consideration for their sensitivities, or influence coefficients. Since these are all random uncertainties, there is, by definition, no correlation in their corresponding error data so these can always be treated as independent uncertainty sources.

Systematic

The systematic uncertainty of the result, B_R, is the root-sum-square of the elemental systematic uncertainties with due consideration for those that are correlated [7]. The general equation is

$$B_R = \left\{ \sum_{T=A}^{B} \sum_{i=1}^{N_T} \left[\left(\theta_i B_{i,T}\right)^2 + \sum_{j=1}^{N_T} \theta_i \theta_j B_{(i,T),(j,T)} \left(1 - \delta_{i,j}\right) \right] \right\}^{1/2} \tag{4.17}$$

where: $B_{i,T}$ = the i^{th} parameter elemental systematic uncertainty of Type T

B_R = the systematic uncertainty of the measurement or test result

N = the total number of systematic uncertainties

θ_i = the sensitivity of the test or measurement result to the i^{th} systematic uncertainty

θ_j = the sensitivity of the test or measurement result to the j^{th} systematic uncertainty

$B_{(i,T),(j,T)}$ = the covariance of B_i on B_j

$$= \sum_{i=1}^{M} B_{i,T}(l) B_{j,T}(l) \tag{4.18}$$

= the sum of the products of the elemental systematic uncertainties that arise from a common source (l)

l = an index or counter for common uncertainty sources

$\delta_{i,j}$ = the Kronecker delta. $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$ if not [7]

T = an index or counter for the ISO uncertainty type, A or B

Here, each $B_{i,T}$ and $B_{j,T}$ are estimated as $2S_X$ for an assumed normal distribution of errors at 95% confidence with infinite degrees of freedom [6].

The random uncertainty, Equation 4.16, and the systematic uncertainty, Equation 4.17, must be combined to obtain a total uncertainty:

$$U_{R,ENG} = t_{95} \left[\left(B_R/2\right)^2 + \left(S_{\bar{X},R}\right)^2 \right]^{1/2} \tag{4.19}$$

Note that B_R is in units of the test or measurement result as was $S_{\bar{X},R_i}$.

The degrees of freedom will be needed for the engineering system total uncertainty. It is accomplished with the Welch–Satterthwaite approximation, the general form of which is Equation 4.10, and the specific formulation here is

$$df_R = \nu_R = \frac{\left\{ \sum_{T=A}^{B} \left[\sum_{i=1}^{N_{S_{\bar{X}_{i,T}}}} \left(\theta_i S_{\bar{X}_{i,T}}\right)^2 + \sum_{i=1}^{N_{B_{i,T}}} \left(\theta_i B_{i,T/t}\right)^2 \right] \right\}^2}{\left\{ \sum_{T=A}^{B} \left[\sum_{i=1}^{N_{S_{\bar{X}_{i,T}}}} \frac{\left(\theta_i S_{\bar{X}_{i,T}}\right)^4}{\left(\nu_{i,T}\right)} + \sum_{i=1}^{N_{B_{i,T}}} \frac{\left(\theta_i B_{i,T/t}\right)^4}{\left(\nu_{i,T}\right)} \right] \right\}} \tag{4.20}$$

where $N_{S_{\bar{X}_{i,T}}}$ = the number of random uncertainties of Type T
$N_{B_{i,T}}$ = the number of systematic uncertainties of Type T
$\nu_{i,T}$ = the degrees of freedom for the i^{th} uncertainty of Type T
$\nu_{i,T}$ = infinity for all systematic uncertainties
t = Student's t associated with the d.f. for each B_i

Symmetrical Systematic Uncertainties

Most times, all elemental uncertainties will be symmetrical. That is, their ± limits about the measured average will be the same. That is, they will be ±3°C or ±2.05 kPa and the like and not +2.0°C, –1.0°C or, +1.5 kPa, –0.55 kPa. The symmetrical measurement uncertainty may therefore be calculated as follows. (For an elegant treatment of nonsymmetrical uncertainties, see that section in Reference [6].)

Note that throughout these uncertainty calculations, all the uncertainties are expressed in engineering units. All the equations will work with relative units as well. That approach may be seen in Reference [6] also. However, it is often easier to express all the uncertainties and the uncertainty estimation calculations in engineering units and then, at the end, with the total uncertainty, convert the result into relative terms. That is what this section recommends.

4.3 Calculation of Total Uncertainty

ISO Total (Expanded) Uncertainty

The ISO total uncertainty for independent uncertainty sources (the most common) is Equation 4.13:

$$U_{R,\mathrm{ISO}} = \pm t_{95}\left[\left(U_{\mathrm{A}}\right)^2 + \left(U_{\mathrm{B}}\right)^2\right]^{1/2} \tag{4.21}$$

where: $U_{R,\mathrm{ISO}}$ = the measurement uncertainty of the result
U_{A} = the Type A uncertainty for the result
U_{B} = the Type B uncertainty for the result
t_{95} = Student's t_{95} is the recommended multiplier to assure 95% confidence

The ISO uncertainty with some nonindependent uncertainty sources is Equation 4.14:

$$U_{R,\mathrm{ISO}} = \left\{\sum_{T=\mathrm{A}}^{\mathrm{B}}\sum_{i=1}^{N_{i,\mathrm{T}}}\left[\left(\theta_i U_{i,\mathrm{T}}\right)^2 + \sum_{j=1}^{N_{j,\mathrm{T}}}\theta_i\theta_j U_{(i,\mathrm{T}),(j,\mathrm{T})}\left(1-\delta_{i,j}\right)\right]\right\}^{1/2} \tag{4.22}$$

Engineering System Total Uncertainty

The engineering system equation for total uncertainty for independent uncertainty sources (the most common) is

$$U_{R,\mathrm{ENG}} = \pm t_{95}\left[\left(B_R/2\right)^2 + \left(S_{\bar{X},R}\right)^2\right]^{1/2} \tag{4.23}$$

Here, just the first term of Equation 4.23 is needed as all the systematic uncertainty sources are independent.

The engineering system equation for uncertainty for nonindependent uncertainty sources (those with correlated systematic uncertainties) is also Equation 4.23; but remember to use the full expression for B_R, Equation 4.17:

TABLE 4.2 Temperature Measurement Uncertainties, F

Defined measurement process	Systematic uncertainty, B_i	d.f. for B_i	Standard deviation, $S_{\bar{X},i}$	Number of data points, N_i	Random uncertainty, $S_{\bar{X},i}$	Degrees of freedom, d.f., v_i
Calibration of tc	0.06_B	∞	0.3_A	10	0.095_A	9
Reference junction	0.07_A	12	0.1_A	5	0.045_A	4
Data acquisition	0.10_B	∞	0.6_A	12	0.173_A	11
RSS						

$$B_R = \left\{ \sum_{T=A}^{B} \sum_{i=1}^{N_T} \left[\left(\theta_i B_{i,T}\right)^2 + \sum_{j=1}^{N_T} \theta_i \theta_j B_{(i,T),(j,T)} \left(1 - \delta_{i,j}\right) \right] \right\}^{1/2} \tag{4.24}$$

The degrees of freedom for Equations 4.21 through 4.24 is calculated with the Welch–Satterthwaite approximation, Equation 4.12 for the ISO system and Equation 4.20 for the engineering system.

High Degrees of Freedom Approximation

It is often the case that it is assumed that the degrees of freedom are 30 or higher. In these cases, the equations for uncertainty simplify further by setting t_{95} equal to 2.000. This approach is recommended for a first-time user of uncertainty analysis procedures as it is a fast way to get to an approximation of the measurement uncertainty.

Calculation Example

The following calculation example is taken where all the uncertainties are independent and are in the units of the test result — temperature. It is a simple example that illustrates the combination of measurement uncertainties in their most basic case. More detailed examples are given in many of the references cited. Their review may be needed to assure a more comprehensive understanding of uncertainty analysis.

It has been shown [8] that there is often little difference in the uncertainties calculated with the different models. The data from Table 4.2 [9] will be used to calculate measurement uncertainty with these two models. These data are all in temperature units and thus the influence coefficients, or sensitivities, are all unity.

Note the use of subscripts "A" and "B" to denote where data exist to calculate a standard deviation. Note too that in this example, all errors (and therefore uncertainties) are independent and that all degrees of freedom for the systematic uncertainties are infinity except for the reference junction whose degrees of freedom are 12. Also note that B_R is calculated as:

$$B_R = 2\left[\left(\frac{0.06}{2}\right)^2 + \left(\frac{0.07}{2.18}\right)^2 + \left(\frac{0.1}{2}\right)^2\right]^{1/2} = 0.13 \tag{4.25}$$

Each uncertainty model will now be used to derive a measurement uncertainty.

For the U_{ISO} model one obtains, via Equation 4.13, the expression:

$$U_A = \left[(0.095)^2 + (0.045)^2 + (0.173)^2 + \left(\frac{0.07}{2.18}\right)^2\right]^{1/2} = 0.21 \tag{4.27}$$

$$U_B = \left[\left(\frac{0.06}{2}\right)^2 + \left(\frac{0.10}{2}\right)^2\right]^{1/2} = 0.058 \tag{4.28}$$

$$U_{R,\mathrm{ISO}} = \pm K\left[\left(U_A\right)^2 + \left(U_B\right)^2\right]^{1/2} = \pm K\left[\left(0.21\right)^2 + \left(0.058\right)^2\right]^{1/2} \tag{4.29}$$

Here, remember that the 0.21 is the root sum square of the $1S_{\bar{X}}$ Type A uncertainties in Table 4.2, and 0.058 that for the $1S_{\bar{X}}$ Type B uncertainties. Also note that in most cases, the Type B uncertainties have infinite degrees of freedom and represent an equivalent $2S_X$. That is why they are divided by 2 — to get an equivalent $1S_X$. Where there are less than 30 degrees of freedom, one needs to divide by the appropriate Student's t that gave the 95% confidence interval. For the reference junction systematic uncertainty above, that was 2.18.

If "K" is taken as Student's t_{95}, the degrees of freedom must first be calculated. Remember that all the systematic components of Type "B" have infinite degrees of freedom except for the 0.07, which has 12 degrees of freedom. Also, all the B_i in Table 4.1 represent an equivalent $2S_X$ except for 0.07, which represents $2.18S_X$, as its degrees of freedom are 12 and not infinity. To use their data here, divide them all but the 0.07 by 2 and the 0.07 by 2.18 so they all now represent $1S_X$, as do the random components. All Type A uncertainties, whether systematic or random in Table 4.1, have degrees of freedom as noted in the table. The degrees of freedom for U_{ISO} is then:

$$df_R = \nu_R = \frac{\left[\left(0.095\right)^2 + \left(0.045\right)^2 + \left(0.173\right)^2 + \left(0.06/2\right)^2 + \left(0.07/2.18\right)^2 + \left(0.10/2\right)^2\right]^2}{\left[\frac{\left(0.095\right)^4}{9} + \frac{\left(0.045\right)^4}{4} + \frac{\left(0.173\right)^4}{11} + \frac{\left(0.06\right)^4}{\infty} + \frac{\left(0.07/2.18\right)^4}{12} + \frac{\left(0.10/2\right)^4}{\infty}\right]} = 22.51 \approx 22 \tag{4.30}$$

t_{95} is therefore 2.07. $U_{R,\mathrm{ISO}}$ is then:

$$U_{R,\mathrm{ISO}} = \pm 2.07\left[\left(0.21\right)^2 + \left(0.058\right)^2\right]^{1/2} = 0.45 \text{ for 95\% confidence} \tag{4.31}$$

For a detailed comparison to the engineering system, here denoted as the $U_{R,\mathrm{ENG}}$ model, three significant figures are carried so as not to be affected by round-off errors. Then:

$$U_{R,\mathrm{ISO}} = \pm 2.074\left[\left(0.205\right)^2 + \left(0.0583\right)^2\right]^{1/2} = 0.442 \text{ for 95\% confidence} \tag{4.32}$$

For the engineering system, $U_{R,\mathrm{ENG}}$, model, Equation 4.23, one obtains the expression:

$$U_{R,\mathrm{ENG}} = \pm t_{95}\left[\left(0.13/2\right)^2 + \left(0.20\right)^2\right]^{1/2} \tag{4.33}$$

Here, the (0.13/2) is the $B_R/2$ and the 0.20 is as before the random component. To obtain the proper t_{95}, the degrees of freedom need to be calculated just as in Equation 4.30. There, the degrees of freedom were 22 and t_{95} equals 2.07. $U_{R,\mathrm{ENG}}$ is then:

$$U_{R,\mathrm{ENG}} = \pm 2.07\left[\left(0.13/2\right)^2 + \left(0.20\right)^2\right]^{1/2} = 0.44 \text{ for 95\% confidence} \tag{4.34}$$

Carrying four significant figures for a comparison to $U_{R,ISO}$ not affected by round-off errors, one obtains:

$$U_{R,\mathrm{ENG}} = \pm 2.074\left[\left(0.133/2\right)^2 + \left(0.202\right)^2\right]^{1/2} = 0.442 \text{ for 95\% confidence} \tag{4.35}$$

This is identical to $U_{R,\mathrm{ISO}}$, Equation 4.32, as predicted.

4.4 Summary

Although these formulae for uncertainty calculations will not handle every conceivable situation, they will provide, for most experimenters, a useful estimate of test or measurement uncertainty. For more detailed treatment or specific applications of these principles, consult the references and the recommended "Further Information" section at the end of this chapter.

Defining Terms

Accuracy: The antithesis of uncertainty. An expression of the maximum possible limit of error at a defined confidence.

Confidence: A statistical expression of percent likelihood.

Correlation: The relationship between two datasets. It is not necessarily evidence of cause and effect.

Degrees of freedom: The amount of room left for error. It may also be expressed as the number of independent opportunities for error contributions to the composite error.

Error: [Error] = [Measured] – [True]. It is the difference between the measured value and the true value.

Influence coefficient: *See* sensitivity.

Measurement uncertainty: The maximum possible error, at a specified confidence, that may reasonably occur. Errors larger than the measurement uncertainty should rarely occur.

Non-symmetrical uncertainty: An uncertainty for which there is an uneven likelihood that the true value lies on one side of the average or the other.

Propagation of uncertainty: An analytical technique for evaluating the impact of an error source (and its uncertainty) on the test result. It employs the use of influence coefficients.

Random error: An error that causes scatter in the test result.

Random uncertainty: An estimate of the limits of random error, usually one standard deviation of the average.

Sensitivity: An expression of the influence an error source has on a test or measured result. It is the ratio of the change in the result to an incremental change in an input variable or parameter measured.

Standard deviation of the average or mean: The standard deviation of the data divided by the number of measurements in the average.

Systematic error: An error that is constant for the duration of a test or measurement.

Systematic uncertainty: An estimate of the limits of systematic error, usually taken as 95% confidence for an assumed normal error distribution.

True value: The desired result of an experimental measurement.

Welch–Satterthwaite: The approximation method for determining the number of degrees of freedom in the random uncertainty of a result.

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Further Information

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